

# Graphlike families of multiweights

Agnese Baldisserri, Elena Rubei

## Abstract

Let  $\mathcal{G} = (G, w)$  be a weighted graph, that is, a graph  $G$  endowed with a function  $w$  from the edge set of  $G$  to the set of real numbers; for any subset  $S$  of the vertex set of  $G$ , we define  $D_S(\mathcal{G})$  to be the minimum of the weights of the subgraphs of  $G$  whose vertex set contains  $S$ ; we call  $D_S(\mathcal{G})$  a multiweight of  $\mathcal{G}$ .

Let  $X$  be a finite set and let  $\{D_S\}_{S \subset X, \#S \geq 2}$  be a family of positive real numbers. We find necessary and sufficient conditions for the family to be the family of multiweights of a positive-weighted graph with vertex set  $X$ . Moreover we study the analogous problem for trees. Finally, we find a criterion to say if there exists a nonnegative-weighted tree  $\mathcal{T}$  with leaf set  $X$  and such that  $D_S(\mathcal{T}) = D_S$  for any  $S \subset X$ .

## 1 Introduction

For any graph  $G$ , let  $E(G)$ ,  $V(G)$  and  $L(G)$  be respectively the set of the edges, the set of the vertices and the set of the leaves of  $G$ . A **weighted graph**  $\mathcal{G} = (G, w)$  is a graph  $G$  endowed with a function  $w : E(G) \rightarrow \mathbb{R}$ . For any edge  $e$ , the real number  $w(e)$  is called the weight of the edge; for any subgraph  $G'$  of  $G$ , we denote by  $w(G')$  the sum of the weights of the edges of  $G'$ . If the weights of all the edges of  $G$  are positive (respectively nonnegative), we say that the graph is **positive-weighted** (respectively **nonnegative-weighted**). If the weights of the internal edges are positive we say that the graph is **internal-positive-weighted**, where an edge  $e$  is said internal if there exists a path with endpoints of degree greater than 2 and containing  $e$ .

Throughout the paper we will consider only simple finite connected graphs.

**Definition 1.** Let  $\mathcal{G} = (G, w)$  be a positive-weighted graph. For any  $k$ -subset  $S$  of  $V(G)$  with  $k \geq 2$ , we define

$$D_S(\mathcal{G}) = \min\{w(R) \mid R \text{ a connected subgraph of } G \text{ such that } S \subset V(R)\}.$$

We call  $D_S(\mathcal{G})$  a **multiweight** of  $\mathcal{G}$  or, more precisely, a  $k$ -weight of  $\mathcal{G}$ . If  $R$  is a connected subgraph of  $G$  such that  $S \subset V(R)$  and  $w(R) = D_S(\mathcal{G})$ , we say that  $R$  realizes  $D_S(\mathcal{G})$ ; observe that  $R$  is necessarily a tree. For simplicity, we denote  $D_{\{i_1, \dots, i_k\}}(\mathcal{G})$  by  $D_{i_1, \dots, i_k}(\mathcal{G})$ .

---

**2010 Mathematical Subject Classification:** 05C05, 05C12, 05C22

**Key words:** weighted graphs, dissimilarity families

We can wonder when a family of positive real numbers is the family of multiweights of some graph or some tree. Let  $n \in \mathbb{N}_{\geq 2}$  (the set of the natural numbers greater than or equal to 2) and  $A$  be a subset of  $\{S \subset \{1, \dots, n\} \mid \#S \geq 2\}$ . A family of positive real numbers  $\{D_S\}_{S \in A}$  is said **p-graphlike** (respectively **nn-graphlike**, **ip-graphlike**) if there exists a positive-weighted (respectively nonnegative, internal-positive) graph  $\mathcal{G} = (G, w)$  with  $\{1, \dots, n\} \subset V(G)$  such that  $D_S(\mathcal{G}) = D_S$  for any  $S \in A$ . If so, we say that  $\mathcal{G}$  **realizes** the family  $\{D_S\}_{S \in A}$ . The vertices  $1, \dots, n$  are called **labelled**. In the case the graph is a tree, we speak of **p-treelike** families, **nn-treelike** families, **ip-treelike** families. Finally, if there exists a positive-weighted tree (respectively a nonnegative-weighted tree, internal-positive-weighted tree)  $\mathcal{T} = (T, w)$  realizing the family and such that  $\{1, \dots, n\} \subset L(T)$ , we say that the family is **p-l-treelike** (respectively **nn-l-treelike**, **ip-l-treelike**). Observe that a family of positive real numbers  $\{D_S\}_{S \in A}$  is p-treelike if and only if it is nn-l-treelike.

Weighted graphs have applications in several disciplines, such as biology, psychology, archeology, engineering. Phylogenetic trees are weighted trees whose vertices represent species and the weight of an edge is given by how much the DNA sequences of the species represented by the vertices of the edge differ. Weighted trees are used also to represent the evolution of languages or of manuscripts. Weighted graphs can represent hydraulic webs or railway webs where the weight of a line is the difference between the earnings and the cost of the line or the length of the line. It can be interesting, given a family  $\mathcal{F}$  of real numbers, to wonder if there exists a weighted tree or, more generally, a weighted graph, with  $\mathcal{F}$  as family of multiweights.

There are several results about families of  $k$ -weights of weighted graphs or trees with fixed  $k$ . One of the first is due to Hakimi and Yau: in 1965, they observed that a family of positive real numbers,  $\{D_I\}_{I \subset \{1, \dots, n\}, \#I=2}$ , is p-graphlike if and only if the  $D_I$  satisfy the triangle inequalities (see [9]). In the same years, also a criterion for a family  $\{D_I\}_{I \subset \{1, \dots, n\}, \#I=2}$  to be p-treelike was established, see [6], [13], [15]:

**Theorem 2.** *Let  $\{D_I\}_{I \subset \{1, \dots, n\}, \#I=2}$  be a family of positive real numbers satisfying the triangle inequalities. It is p-treelike (or nn-l-treelike) if and only if it satisfies the so-called 4-point condition: for all  $a, b, c, d \in \{1, \dots, n\}$ , the maximum of*

$$\{D_{a,b} + D_{c,d}, D_{a,c} + D_{b,d}, D_{a,d} + D_{b,c}\}$$

*is attained at least twice.*

For higher  $k$  the literature is more recent. In [12] Pachter and Speyer explained that the study of  $k$ -weights, with  $k > 2$ , is important because they are statistically more reliable than 2-weights (see also [14]). Moreover, they obtained an important result about  $k$ -weights of positive-weighted trees with  $n$  labeled leaves and  $k \leq \frac{n+1}{2}$ :

**Theorem 3. (Pachter-Speyer).** *Let  $k, n \in \mathbb{N}$  with  $3 \leq k \leq \frac{n+1}{2}$ . A positive-weighted tree  $\mathcal{T}$  with leaves  $1, \dots, n$  and no vertices of degree 2 is determined by the values  $D_I(\mathcal{T})$ , where  $I \subset \{1, \dots, n\}, \#I = k$ .*

Later, the study of the families of  $k$ -weights of weighted trees produced several other results, see for example [8], [11], [2] or [3].

The results we have quoted are about families of  $k$ -weights with fixed  $k$ , but it can be interesting also to characterize the families of multiweights, that is the families of  $k$ -weights with  $k$  varying in

$\mathbb{N}_{\geq 2}$ , of weighted graphs or trees. Some results in this sense are due to Bryant and Tupper, who, in [4] and in [5], discovered important properties regarding the concept of diversity:

**Definition 4.** A **diversity** is a pair  $(X, \delta)$  where  $X$  is a set and  $\delta$  is a function from the finite subsets of  $X$  to  $\mathbb{R}$  satisfying the following conditions:

- (1)  $\delta(A) \geq 0$  for any finite  $A \subset X$ ; moreover  $\delta(A) = 0$  if and only if  $\#A \leq 1$ ;
- (2) if  $B \neq \emptyset$ , then  $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$  for all finite  $A, B, C \subset X$ .

It is easy to prove that conditions (1) and (2) imply that, if  $A \subset B$ , then  $\delta(A) \leq \delta(B)$ . Obviously, if  $\mathcal{G} = (G, w)$  is a positive-weighted graph and we consider the pair  $(\mathcal{P}, \delta)$  where

$$\mathcal{P} = \{S \subset V(G) \mid \#S \geq 2\}$$

and, for any  $S \in \mathcal{P}$ ,

$$\delta(S) = D_S(\mathcal{G}),$$

we have that  $(\mathcal{P}, \delta)$  is a diversity.

In this paper we study families of  $k$ -weights of positive-weighted graphs with  $k$  varying in  $\mathbb{N}_{\geq 2}$ . Precisely, let

$$\{D_S\}_{S \subset \{1, \dots, n\}, \#S \geq 2}$$

be a family of positive real numbers; we find necessary and sufficient conditions for it to be the family of multiweights of a positive weighted-graph or a positive-weighted tree with vertex set  $\{1, \dots, n\}$ , see respectively Theorem 9 and Theorem 12. Our results are based on a proposition (Proposition 8) that relates the  $k$ -weights of a positive-weighted graph or tree for  $k \geq 3$  to the 2-weights. Finally we study the analogous problem for nonnegative-weighted trees with set of leaves equal to  $\{1, \dots, n\}$  (see Theorem 19).

## 2 Notation

**Notation 5.** • Throughout the paper, let  $n \in \mathbb{N}$  with  $n \geq 2$ ; we denote by  $[n]$  the set  $\{1, \dots, n\}$ .

- For any set  $S$  and  $k \in \mathbb{N}$ , let  $\binom{S}{k}$  be the set of the  $k$ -subsets of  $S$  and let  $\binom{S}{\geq k}$  be the set of the subsets of  $S$  of cardinality greater than or equal to  $k$ .
- The words “graph” and “tree” denote respectively a finite graph and a finite tree.
- Let  $\{D_I\}_{I \in A}$  with  $A \subset \binom{[n]}{\geq 2}$  be a family of real numbers. For simplicity, we denote  $D_{\{i_1, \dots, i_k\}}$  by  $D_{i_1, \dots, i_k}$ .

**Notation 6.** Let  $T$  be a tree.

- A **node** of  $T$  is a vertex of degree greater than 2.
- Let  $F$  be a leaf of  $T$ . Let  $N$  be the node such that the path  $p$  between  $N$  and  $F$  does not contain any node apart from  $N$ . We say that  $p$  is the **twig** associated to  $F$ . We say that an edge is **internal** if it is not an edge of a twig. It is easy to see that this definition is equivalent to the one we have given in the introduction. We denote by  $\dot{E}(T)$  the set of the internal edges of  $T$ .
- We say that  $T$  is **essential** if it has no vertices of degree 2.
- If  $a$  and  $b$  are vertices of  $T$ , we denote by  $p(a, b)$  the path between  $a$  and  $b$ .

- Let  $S$  be a subset of  $L(T)$ . We denote by  $T|_S$  the minimal subtree of  $T$  whose vertex set contains  $S$ . If  $\mathcal{T} = (T, w)$  is a weighted tree, we denote by  $\mathcal{T}|_S$  the tree  $T|_S$  with the weighting induced by  $w$ . Let  $\tilde{E}(T|_S) = \mathring{E}(T) \cap E(T|_S)$ . Observe that in general  $\tilde{E}(T|_S) \neq \mathring{E}(T|_S)$ , see Figure 1 for an example.

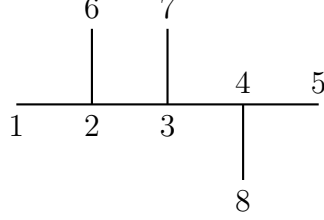


Figure 1: Let  $T$  be the tree in the figure and let  $S = \{1, 5, 7, 8\}$ . The edge  $\{2, 3\}$  is in  $\tilde{E}(T|_S)$  but not in  $\mathring{E}(T|_S)$ .

- We say that two leaves  $i$  and  $j$  of  $T$  are **neighbours** if in  $p(i, j)$  there is only one node; furthermore, we say that  $C \subset L(T)$  is a **cherry** if any  $i, j \in C$  are neighbours. The **stalk** of a cherry is the unique node in the path with endpoints any two elements of the cherry.
- Let  $a, b, c, d \in L(T)$ . We say that  $\langle a, b | c, d \rangle$  holds if in  $T|_{\{a, b, c, d\}}$  we have that  $a$  and  $b$  are neighbours,  $c$  and  $d$  are neighbours, and  $a$  and  $c$  are not neighbours; in this case we denote by  $\gamma_{a, b, c, d}$  the path between the stalk  $s_{a, b}$  of  $\{a, b\}$  and the stalk  $s_{c, d}$  of  $\{c, d\}$  in  $T|_{\{a, b, c, d\}}$ ; we call it the **bridge** of the 4-subset  $\{a, b, c, d\}$ . The symbol  $\langle a, b | c, d \rangle$  is called **Buneman's index** of  $a, b, c, d$ .

### 3 Graphs and trees with all the vertices labelled

**Definition 7.** For any  $X \subset \binom{[n]}{2}$ , we define  $G_X$  to be the graph such that  $E(G_X) = X$  and  $V(G_X) = \cup_{I \in X} I$ .

**Proposition 8.** Let  $\mathcal{G} = (G, w)$  be a positive-weighted graph with  $V(G) = [n]$ . For any  $S \in \binom{[n]}{\geq 3}$ , we have:

$$D_S(\mathcal{G}) = \min_{\substack{X \subset \binom{[n]}{2} \text{ s.t. } G_X \text{ tree} \\ \text{and } L(G_X) \subset S \subset V(G_X)}} \left\{ \sum_{I \in X} D_I(\mathcal{G}) \right\}.$$

*Proof.* Let us fix  $S \in \binom{[n]}{\geq 3}$  and define

$$R = \left\{ X \subset \binom{[n]}{2} \mid G_X \text{ a tree and } L(G_X) \subset S \subset V(G_X) \right\}.$$

Let  $T$  be a connected subtree of  $G$  realizing  $D_S(\mathcal{G})$ . Then, obviously,

$$D_S(\mathcal{G}) = w(T) = \sum_{I \in E(T)} w(I). \quad (1)$$

We want to prove that

$$w(I) = D_I(\mathcal{G}) \quad (2)$$

for any  $I \in E(T)$ . Obviously,  $w(I) \geq D_I(\mathcal{G})$ ; moreover, if, contrary to our claim, we had that  $w(I) > D_I(\mathcal{G})$  for some  $I \in E(T)$ , then it would exist a path  $p_I$  from one element of  $I$  to the other, different from the edge  $I$ , and such that  $D_I(\mathcal{G}) = w(p_I)$ ; let  $H$  be the connected subgraph of  $G$  obtained from  $T$  replacing the edge  $I$  with  $p_I$ ; we would have that  $S \subset V(H)$  and  $w(H) < w(T) = D_S(\mathcal{G})$ , which is absurd. From (1) and (2) we get

$$D_S(\mathcal{G}) = \sum_{I \in E(T)} D_I(\mathcal{G}).$$

From the equation above and the fact that  $E(T) \in R$ , we get that

$$D_S(\mathcal{G}) \geq \min_{X \in R} \left\{ \sum_{I \in X} D_I(\mathcal{G}) \right\}.$$

Let us prove the other inequality; suppose  $X \in R$ ; for any  $I \in X$ , let  $p_I$  be a path in  $\mathcal{G}$  realizing  $D_I(\mathcal{G})$ . We have that

$$D_S(\mathcal{G}) \leq w \left( \bigcup_{I \in X} p_I \right) \leq \sum_{I \in X} w(p_I) = \sum_{I \in X} D_I(\mathcal{G}),$$

where the first inequality holds because  $\bigcup_{I \in X} p_I$  is a connected graph and its vertex set contains  $S$ .  $\square$

**Theorem 9.** *Let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers. There exists a positive-weighted graph  $\mathcal{G} = (G, w)$ , with  $V(G) = [n]$ , such that  $D_I(\mathcal{G}) = D_I$  for any  $I \in \binom{[n]}{\geq 2}$  if and only if the following two conditions hold:*

- (i)  $D_{i,j} \leq D_{i,k} + D_{j,k}$  for any  $i, j, k \in [n]$ ;
- (ii) for any  $S \in \binom{[n]}{\geq 3}$  we have that

$$D_S = \min_{\substack{X \subset \binom{[n]}{2} \text{ s.t. } G_X \text{ tree} \\ \text{and } L(G_X) \subset S \subset V(G_X)}} \left\{ \sum_{I \in X} D_I \right\}.$$

*Proof.*  $\implies$  Suppose that there exists a positive-weighted graph  $\mathcal{G} = (G, w)$ , with  $V(G) = [n]$ , such that  $D_I(\mathcal{G}) = D_I$  for any  $I \in \binom{[n]}{\geq 2}$ . It is well known and easy to prove that condition (i) holds. Condition (ii) follows from Proposition 8.

$\impliedby$  Let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers satisfying conditions (i) and (ii); we can construct a positive-weighted graph  $\mathcal{G} = (G, w)$  in the following way: let  $G$  be the complete graph with  $n$  vertices, and let the weight of the edge  $\{i, j\}$  be equal to  $D_{i,j}$ . We have that  $D_I(\mathcal{G}) = D_I$

for any  $I \in \binom{[n]}{2}$  by (i). We have to prove that  $D_S(\mathcal{G}) = D_S$  for any  $S \in \binom{[n]}{\geq 3}$ . By Proposition 8 we know that

$$D_S(\mathcal{G}) = \min_{X \in R} \left\{ \sum_{I \in X} D_I(\mathcal{G}) \right\},$$

where  $R = \{X \subset \binom{[n]}{2} \mid G_X \text{ a tree and } L(G_X) \subset S \subset V(G_X)\}$ , and by assumption (ii) we have that

$$D_S = \min_{X \in R} \left\{ \sum_{I \in X} D_I \right\}.$$

So we get the desired result, since we have already proved that  $D_I(\mathcal{G}) = D_I$  for any  $I \in \binom{[n]}{2}$ .  $\square$

Now, we want to characterize the families of multiweights of positive-weighted trees. First, we need to introduce a definition and to state a theorem characterizing the families  $\{D_I\}_{I \in \binom{[n]}{2}}$  that are the families of 2-weights of positive-weighted trees with  $[n]$  as vertex set (that is, with all the vertices labelled). The theorem, probably well-known to experts, was suggested to us by an anonymous referee in October 2014 as a simplification of a similar but more complicated criterion; later we have found it also in [10]; we give here a shorter proof.

**Definition 10.** Let  $n \geq 3$  and let  $\{D_I\}_{I \in \binom{[n]}{2}}$  be a set of positive real numbers. We say that the family  $\{D_I\}$  is a **median family** if, for any  $a, b, c \in [n]$ , there exists a unique element  $m \in [n]$  such that

$$D_{i,j} = D_{i,m} + D_{j,m}$$

for any distinct  $i, j \in \{a, b, c\}$ .

Observe that a median family satisfies the triangle inequalities.

**Theorem 11.** Let  $n \geq 3$  and let  $\{D_I\}_{I \in \binom{[n]}{2}}$  be a family of positive real numbers. There exists a positive-weighted tree  $\mathcal{T} = (T, w)$ , with  $V(T) = [n]$ , such that  $D_I(\mathcal{T}) = D_I$  for all  $I \in \binom{[n]}{2}$  if and only if the 4-point condition holds and the family  $\{D_I\}_I$  is median.

*Proof.*  $\implies$  Obvious.

$\impliedby$  Since the 4-point condition holds, then, by Theorem 2, there exists a positive-weighted tree  $\mathcal{T} = (T, w)$  such that  $[n] \subset V(T)$  and  $D_I(\mathcal{T}) = D_I$  for all  $I \in \binom{[n]}{2}$ . Obviously we can suppose that the vertices of  $T$  of degree 1 or 2 are elements of  $[n]$ . We want to prove that  $V(T) = [n]$ . Let  $m \in V(T)$  such that  $\deg(m) \geq 3$ . This implies that there exist three distinct leaves,  $a, b, c \in [n]$ , such that  $T|_{\{a,b,c\}}$  is a star with center the vertex  $m$ . We have that:

$$D_{i,j}(\mathcal{T}) = D_{i,m}(\mathcal{T}) + D_{j,m}(\mathcal{T})$$

for any  $i, j \in \{a, b, c\}$ . Moreover, by assumption, there exists a unique element  $z \in [n]$  such that:

$$D_{i,j} = D_{i,z} + D_{j,z} \tag{3}$$

for any  $i, j \in \{a, b, c\}$ , that is,

$$D_{i,j}(\mathcal{T}) = D_{i,z}(\mathcal{T}) + D_{j,z}(\mathcal{T})$$

for any  $i, j \in \{a, b, c\}$ .

Then,  $z = m$ ; in fact, if  $z \in V(T|_{\{a,b,c\}})$  and  $z \neq m$ , then  $z$  would be a vertex of the path between  $m$  and one of the three leaves, suppose for example  $a$ ; this would imply that  $D_{b,c} \neq D_{b,z} + D_{c,z}$ , which is absurd by (3). On the other hand, if  $z \notin V(T|_{\{a,b,c\}})$ , similarly we would obtain that  $D_{i,j} \neq D_{i,z} + D_{j,z}$  for any  $i, j \in \{a, b, c\}$ , which is absurd. So  $z = m$ ; therefore  $m \in [n]$ , as we wanted to prove.  $\square$

**Theorem 12.** *Let  $n \geq 3$  and let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers. There exists a positive-weighted tree  $\mathcal{T} = (T, w)$ , with  $V(T) = [n]$ , such that  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{[n]}{\geq 2}$  if and only if the following three conditions hold:*

- (i) *the family  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  satisfies the 4-point condition;*
- (ii) *the family  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  is median;*
- (iii) *for any  $S \in \binom{[n]}{\geq 3}$ , we have:*

$$D_S = \min_{\substack{X \subset \binom{[n]}{2} \text{ s.t. } G_X \text{ tree} \\ \text{and } L(G_X) \subset S \subset V(G_X)}} \left\{ \sum_{I \in X} D_I \right\}. \quad (4)$$

*Proof.*  $\implies$  Condition (i) is satisfied by Theorem 2, condition (ii) is satisfied by Theorem 11 and, finally, Proposition 8 assures us of the last condition.

$\Leftarrow$  Let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers satisfying conditions (i), (ii) and (iii). Suppose that  $\mathcal{T} = (T, w)$  is a positive-weighted tree such that  $V(T) = [n]$  and  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{[n]}{2}$  (such a tree exists by Theorem 11 and conditions (i) and (ii)); we can prove that  $D_S(\mathcal{T}) = D_S$  for any  $S \in \binom{[n]}{\geq 3}$  arguing as in the proof of Theorem 9.  $\square$

## 4 Trees with labels only on the leaves

In order to study the families of multiweights of nonnegative-weighted trees with set of leaves equal to  $[n]$ , we need to recall some notation and some facts from [1].

**Definition 13.** *Let  $T$  be a tree and let  $a, b, c, d, x \in L(T)$ . Let  $S$  be a subtree of  $T|_{a,b,c,d}$ .*

*Let  $\tilde{x}$  be the vertex such that  $p(x, \tilde{x})$  is the minimal path whose union with  $T|_{a,b,c,d}$  is connected; we say that  $x$  **clings** to  $T|_{a,b,c,d}$  in  $S$  if  $\tilde{x} \in V(S)$ .*

See Figure 2 for an example: let  $T$  be the tree in the figure and let  $S = p(a, b)$ .

**Definition 14.** *Let  $\{D_I\}_{I \in \binom{[n]}{2}}$  be a family of real numbers. For any distinct  $a, b, c, d \in [n]$ , let*

$$L_{\{a,b\}}^{\{a,b,c,d\}} = \left\{ x \in [n] - \{a, b, c, d\} \mid \begin{array}{l} \text{either } D_{x,z} - D_{a,z} \text{ does not depend on } z \in \{b, c, d\} \\ \text{or } D_{x,z} - D_{b,z} \text{ does not depend on } z \in \{a, c, d\} \end{array} \right\} \cup \{a, b\}.$$

*We will denote  $L_{\{a,b\}}^{\{a,b,c,d\}}$  simply by  $L_{a,b}^{a,b,c,d}$  and we will omit the superscript when the 4-set which we are referring to is clear from the context.*

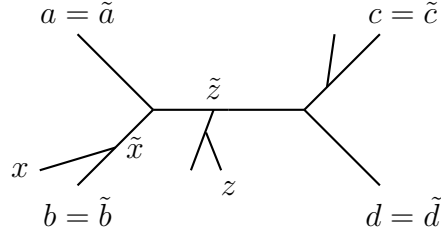


Figure 2: the leaves  $x, a, b$  cling to  $T|_{a,b,c,d}$  in  $S := p(a, b)$ , while  $z, c, d$  do not cling to  $T|_{a,b,c,d}$  in  $S$

The definition above seems rather obscure, but the following proposition and example will clarify it.

**Proposition 15.** *Let  $\mathcal{T} = (T, w)$  be an essential internal-positive-weighted tree. Denote  $D_{i,j}(\mathcal{T})$  by  $D_{i,j}$  for distinct  $i, j \in L(T)$ . Let  $a, b, c, d \in L(T)$ .*

- 1) *If  $\langle a, b | c, d \rangle$  holds, we have that  $L_{a,b}^{a,b,c,d}$  is the set of the elements  $x$  of  $L(T)$  clinging to  $T|_{a,b,c,d}$  in  $p(a, b)$  and  $L_{c,d}^{a,b,c,d}$  is the set of the elements  $x$  of  $L(T)$  clinging to  $T|_{a,b,c,d}$  in  $p(c, d)$ .*
- 2) *We have that  $\langle a, b | c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given by exactly one edge if and only if the following conditions hold:*

(i)

$$D_{a,b} + D_{c,d} < D_{a,c} + D_{b,d} = D_{a,d} + D_{b,c};$$

(ii)  $L_{a,b} \cup L_{c,d} = L(T)$ .

See [1] for the proof.

**Example.** Let  $\mathcal{T}$  be the tree represented in Figure 3 with all the weights of the edges equal to 1; consider the 4-set  $\{1, 3, 4, 7\}$ ; we have that  $\langle 1, 3 | 4, 7 \rangle$  holds,  $L_{1,3}^{1,3,4,7} = \{1, 2, 3, 9, 10\}$  and  $L_{4,7}^{1,3,4,7} = \{4, 5, 6, 7, 8\}$ , so  $L_{1,3}^{1,3,4,7} \cup L_{4,7}^{1,3,4,7} = [10]$  and  $\gamma_{1,3,4,7}$  is composed by only one edge. Now consider the 4-set  $\{1, 9, 4, 7\}$ ; we have that  $\langle 1, 9 | 4, 7 \rangle$  holds,  $L_{1,9}^{1,9,4,7} = \{1, 2, 9, 10\}$  and  $L_{4,7}^{1,9,4,7} = \{4, 5, 6, 7, 8\}$ , so  $L_{1,9}^{1,9,4,7} \cup L_{4,7}^{1,9,4,7} \neq [10]$  and in fact  $\gamma_{1,9,4,7}$  is composed by more than one edge.

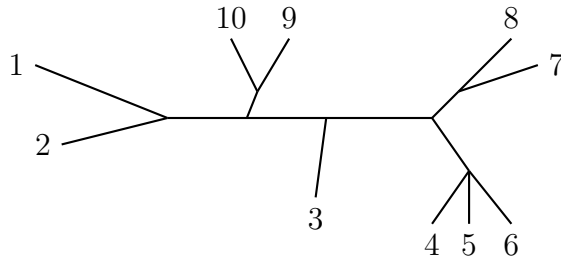


Figure 3: An example to explain Proposition 15



**Definition 16.** Let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers. Let us define

$$Q = \left\{ (a, b, c, d) \text{ ordered 4-subset of } [n] \mid \begin{array}{l} D_{a,b} + D_{c,d} < D_{a,c} + D_{b,d} = D_{a,d} + D_{b,c}, \\ L_{a,b} \cup L_{c,d} = [n] \end{array} \right\} / \sim,$$

where  $(a, b, c, d) \sim (a', b', c', d')$  if and only if  $\{L_{a,b}, L_{c,d}\} = \{L_{a',b'}, L_{c',d'}\}$ .

Moreover, for any  $S \in \binom{[n]}{\geq 4}$ , let

$$Q(S) = \{(a, b, c, d) \in Q \mid S \cap L_{a,b} \neq \emptyset, S \cap L_{c,d} \neq \emptyset\}.$$

If  $[(a, b, c, d)] \in Q(S)$ , we define:

$$q(S)_{[(a,b,c,d)]} = \#(L_{\{a,b\}}^{\{a,b,c,d\}} \cap S) \cdot \#(L_{\{c,d\}}^{\{a,b,c,d\}} \cap S).$$

**Remark 17.** Let  $\mathcal{T} = (T, w)$  be an essential internal-positive-weighted tree with  $L(T) = [n]$ . Denote  $D_{i,j}(\mathcal{T})$  by  $D_{i,j}$  for distinct  $i, j \in L(T)$ . Observe that, by Proposition 15, the set  $Q$  is in bijection with  $\tilde{E}(T)$  and, for any  $S \in \binom{[n]}{\geq 4}$ , the set  $Q(S)$  is in bijection with  $\tilde{E}(T|_S)$ .

**Proposition 18.** Let  $\mathcal{T} = (T, w)$  be a nonnegative-weighted tree with  $L(T) = [n]$ . Let us denote  $D_{i,j}(\mathcal{T})$  by  $D_{i,j}$  for distinct  $i, j \in L(T)$ . For any  $S \subset [n]$  we have that

- if  $\#S = 3$  then

$$D_S(\mathcal{T}) = \frac{1}{2} \sum_{\{i,j\} \in \binom{S}{2}} D_{i,j}; \quad (5)$$

- if  $\#S \geq 4$  then

$$D_S(\mathcal{T}) = \frac{1}{\#S - 1} \left[ \sum_{\{i,j\} \in \binom{S}{2}} D_{i,j} - \sum_{[(a,b,c,d)] \in Q(S)} \frac{D_{a,c} + D_{b,d} - D_{a,b} - D_{c,d}}{2} (q(S)_{[(a,b,c,d)]} + 1 - \#S) \right], \quad (6)$$

where  $Q(S)$  and  $q(S)_{[(a,b,c,d)]}$  are defined as in Definition 16.

*Proof.* Obviously we can suppose that  $T$  is essential and internal-positive-weighted. If  $\#S = 3$ , the minimal subtree of  $T$  containing the elements of  $S$  is a star with three leaves, so it is easy to check (5). If  $\#S \geq 4$ , for any  $i \in S$ , call  $e_i$  the corresponding twig, which is composed by only one edge because  $T$  is essential; by definition we have that

$$D_S(\mathcal{T}) = \sum_{i \in S} w(e_i) + \sum_{e \in \tilde{E}(T|_S)} w(e);$$

thus, by Remark 17,

$$D_S(\mathcal{T}) = \sum_{i \in S} w(e_i) + \sum_{[a,b,c,d] \in Q(S)} \frac{D_{a,c} + D_{b,d} - D_{a,b} - D_{c,d}}{2}. \quad (7)$$

Moreover

$$\sum_{\{i,j\} \in \binom{S}{2}} D_{i,j} = (\#S - 1) \sum_{i \in S} w(e_i) + \sum_{[a,b,c,d] \in Q(S)} q(S)_{[a,b,c,d]} \frac{D_{a,c} + D_{b,d} - D_{a,b} - D_{c,d}}{2},$$

because  $q|_{[(a,b,c,d)]}$  is the number of the 2-subsets  $\{i, j\}$  of  $S$  such that  $i$  and  $j$  belong to different connected components of  $T \setminus \{e\}$  (that is,  $e$  belongs to the path  $p(i, j)$  realizing  $D_{i,j}$ ); hence

$$\sum_{i \in S} w(e_i) = \frac{1}{(\#S - 1)} \sum_{\{i,j\} \in \binom{S}{2}} D_{i,j} - \sum_{[a,b,c,d] \in Q(S)} q(S)_{[a,b,c,d]} \frac{D_{a,c} + D_{b,d} - D_{a,b} - D_{c,d}}{2(\#S - 1)}. \quad (8)$$

From (7) and (8) we get immediately our statement.  $\square$

**Theorem 19.** *Let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers. There exists a nonnegative-weighted tree  $\mathcal{T} = (T, w)$ , with  $L(T) = [n]$ , such that  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{[n]}{\geq 2}$  if and only if the following three conditions hold:*

(i) *the family  $\{D_I\}_{I \in \binom{[n]}{2}}$  satisfies the 4-point condition;*

(ii) *for any  $S \in \binom{[n]}{3}$ , we have:*

$$D_S = \frac{1}{2} \sum_{\{i,j\} \in \binom{S}{2}} D_{i,j}; \quad (9)$$

(iii) *for any  $S \in \binom{[n]}{\geq 4}$ , we have:*

$$D_S = \frac{1}{\#S - 1} \left[ \sum_{\{i,j\} \in \binom{S}{2}} D_{i,j} - \sum_{[a,b,c,d] \in Q(S)} \frac{D_{a,c} + D_{b,d} - D_{a,b} - D_{c,d}}{2} (q(S)_{[(a,b,c,d)]} + 1 - \#S) \right], \quad (10)$$

where  $Q(S)$  and  $q(S)_{[(a,b,c,d)]}$  are defined as in Definition 16.

*Proof.*  $\implies$  Condition (i) is satisfied by Theorem 2 and Proposition 18 assures us of (ii) and (iii).

$\Leftarrow$  Suppose  $\{D_I\}$  satisfies conditions (i), (ii) and (iii). Let  $\mathcal{T} = (T, w)$  be a nonnegative-weighted tree such that  $L(T) = [n]$  and  $D(\mathcal{T}) = D_I$  for every  $I \in \binom{[n]}{2}$  (it exists by Theorem 2). By Proposition 18 and conditions (ii) and (iii), we have that  $D_S(\mathcal{T}) = D_S$  for any  $S \in \binom{[n]}{\geq 3}$ .  $\square$

Since a family of positive real numbers is nn-l-treelike if and only if it is p-treelike, Theorem 19 can be reformulated in the following way:

**Theorem 20.** *Let  $\{D_I\}_{I \in \binom{[n]}{\geq 2}}$  be a family of positive real numbers. There exists a positive-weighted tree  $\mathcal{T} = (T, w)$ , with  $[n] \subset V(T)$ , such that  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{[n]}{\geq 2}$  if and only if conditions (i), (ii) and (iii) of Theorem 19 hold.*

# References

- [1] A. Baldisserri, E. Rubei, *A characterization of dissimilarity families of trees*, submitted.
- [2] A. Baldisserri, E. Rubei, *Families of multiweights and pseudostars*, Adv. in Appl. Math., 77 (2016), 86-100
- [3] A. Baldisserri, E. Rubei, *On graphlike  $k$ -dissimilarity vectors*, Ann. Comb., 18 (2014), no. 3, 356-381 .
- [4] D. Bryant, P. Tupper, *Hyperconvexity and tight span theory for diversities*, Advances in Mathematics 231 (2012), no. 6, 3172-3198
- [5] D. Bryant, P. Tupper, *Diversities and the geometry of hypergraphs*, Discrete Math. Theor. Comput. Sci. 16 (2014), no. 2, 1-20
- [6] P. Buneman, *A note on the metric properties of trees*. Journal of Combinatorial Theory Ser. B 17 (1974), 48-50
- [7] A. Dress, K. T. Huber, J. Koolen, V. Moulton, A. Spillner, Basic phylogenetic combinatorics. Cambridge University Press, Cambridge, 2012
- [8] S.Herrmann, K.Huber, V.Moulton, A.Spillner, *Recognizing treelike  $k$ -dissimilarities*. Journal of Classification 29 (2012), no. 3, 321-340
- [9] S.L. Hakimi, S.S. Yau, *Distance matrix of a graph and its realizability*. Quart. Appl. Math. 22 (1965), 305-317
- [10] M. Hayamizu, K.Fukumizu *On minimum spanning tree-like metric spaces* arXiv: 1505.06145
- [11] D. Levy, R. Yoshida, L. Pachter, *Beyond pairwise distances: neighbor-joining with phylogenetic diversity estimates*, Mol. Biol. Evol. 23 (2006), no. 3, 491-498
- [12] L. Pachter, D. Speyer, *Reconstructing trees from subtree weights*. Appl. Math. Lett. 17 (2004), no. 6, 615–621
- [13] J.M.S. Simoes Pereira, *A Note on the Tree Realizability of a distance matrix*. Journal of Combinatorial Theory 6 (1969), 303-310
- [14] D. Speyer, B. Sturmfels *Tropical mathematics*, Math. Mag. 82 (2009), no. 3, 163-173
- [15] K.A. Zaretskii, *Constructing trees from the set of distances between pendant vertices*, Uspehi Matematicheskikh Nauk. 20 (1965), 90-92

**E-mail addresses:** baldisser@math.unifi.it, rubei@math.unifi.it